Exact solutions for the D-dimensional spherical isotropic confined harmonic oscillator

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Abstract

We study the size effect on the energy levels of the D–dimensional isotropic harmonic oscillator confined within a box of radius r_c with impenetrable walls. Two different approaches are used to obtain the energy eigenvalues and eigenfunctions for D=1,2,...,5. In the first we solve the Schrödinger equation exactly. In the second we use a series expansion of the wave function. The numerical results obtained are extremely accurate; these values are reported with 50 decimal places.

1 Introduction

The idea of the spatial confinement in quantum systems has had a growing interest in recent years, due to its potential application in the study and production of artificial atoms in semiconductor materials and of the future circuit devices of nano and molecular size, including the quantum computers.

On the other hand, confined quantum systems have a long history in the modelling of a great number of applications in different areas of Physics and Chemistry as it is shown in the cited review articles [1]. In the decade of the 30's, Michels, De Boer and Bijl [2] proposed the model of a hydrogen atom confined at the centre of a sphere with impenetrable walls and used it to study the effects of extreme pressure on the electronic states of the hydrogen atom. This model

has become one of the most studied in the literature [2]–[12]. The idea of confinement inside spherical boxes has been broadly accepted and it has continued to be used to study the electronic estates of multi-electron atoms subject to extreme pressures [13]–[16].

Another widely studied confined quantum system is the 1-D confined harmonic oscillator [17]–[39]. This system has been used as model for the study of the proton-deuteron transformation as the energy source in dense stars [17]–[18], in the theory of the white dwarfs [19] and in the escape velocity of stars from the galactic or globular cumulus [20]. It has also been used in the study of the specific heat of solids subjected to high pressures [21] and magnetic properties [22]of metals. Also few studies have been made on the transition probabilities and Einstein coefficients for the transitions between different levels of the 1-D confined harmonic oscillator [25], [38]–[39], showing that new allowed transitions appear as a result of the confinement.

However, the harmonic oscillator confined in two and three dimensions has received less attention [40]–[43]. Recently, we have discussed the incidental degeneracies in the 3-D isotropic confined harmonic oscillator [44], and the conditions for the appearance of the incidental and the inter-dimensional degeneracies in the D-dimensional confined harmonic oscillator [45].

In the study of the incidental and the inter-dimensional degeneracies, it is necessary to have accurate energy eigenvalues. For the free (unconfined) situation, the energy eigenvalues are very well known, but this is not true for the confined problem where analytic expressions are not available and solutions must be obtained numerically. The functional forms of the eigenfunctions of the confined harmonic oscillators in 1, 2 and 3 dimensions have been known for some time [17], [18], [25], [35]–[36], [40]. These are given in terms of the confluent hypergeometric functions. In this work we will present two extremely precise methods to obtain the energies of the D-dimensional confined oscillators. The first is an exact method that is based on numerically obtaining the roots of the confluent hypergeometric functions. The second method is based on the development of the wave function in a Taylor series. This method was used with much success previously [10], [39], [42]. We find that the energy eigenvalues calculated by both methods are identical and we report them with 50 significant figures.

The content of this work is as follows: In the section 2 we present the methods used in the obtaining of the energy. In section 3 we present our results including comparison with previous works. Finally, in section 4 we discuss our results and conclusions.

2 Exact solutions

2.1 The one–dimensional confined HO

The Schrödinger equation for the one-dimensional, symmetrically confined harmonic oscillator (in natural units where $m = \omega = \hbar = 1$) is given by

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\psi(x) = E\psi(x), \tag{1}$$

where the energy is in units of $\hbar\omega$ and the unit of the distance is $\sqrt{\hbar/m\omega}$.

The potential energy is a symmetric function of x, therefore the eigenstates have definite parity; odd or even. The exact solutions are well known [17]–[18], [25]–[26], [30], [35] and are obtained in terms of the Kummer or confluent hypergeometric functions [46]

$$\begin{aligned} \psi^+(x) &= Ae^{-x^2/2}\ {}_1F_1\left[\tfrac{1}{4}(1-2E);\tfrac{1}{2};x^2\right], \\ \psi^-(x) &= Be^{-x^2/2}x\ {}_1F_1\left[\tfrac{1}{4}(3-2E);\tfrac{3}{2};x^2\right], \end{aligned} \tag{2}$$

where $+$ and $-$ indicate even and odd parity respectively.

In order for the wave function to be square integrable, the hypergeometric function for the unconfined one-dimensional oscillator must terminate. This requires that there exist some non negative integer n such that

$$E = n + \frac{1}{2}, \quad n = 0,\ 1,\ 2,\ 3, ... \quad . \tag{3}$$

When the harmonic oscillator is symmetrically confined in a box of length $2x_c$ with impenetrable walls, the energy quantization results from the boundary conditions on the wave functions

$$\psi^{\pm}(x = -x_c) = \psi^{\pm}(x = x_c) = 0. \tag{4}$$

The allowed energies are obtained when the successive roots of the following equations are found

$$\begin{array}{l} {}_1F_1\left[\frac{1}{4}\left(1-2E\right);\frac{1}{2};x_c^2\right] = 0, \quad for\ even\ states \\ {}_1F_1\left[\frac{1}{4}\left(3-2E\right);\frac{3}{2};x_c^2\right] = 0, \quad for\ odd\ states \end{array} \tag{5}$$

To determine the energy eigenvalues, it is necessary to solve numerically for one of the boundary conditions (4) because the symmetry of the problem. In this work we found the allowed energies using the Maple computer algebra system and Maple's root-finding function FSOLVE. Our results are reported in Table I.

2.2 The D–dimensional confined HO

The Schrödinger equation for the isotropic harmonic oscillator in a D-dimensional Cartesian coordinate system $x_1, x_2, \ldots, x_D$ is

$$\left(-\frac{1}{2}\Delta^{(D)} + \frac{1}{2}r^2\right)\Psi^{(D)}\left(x_1, x_2, \ldots, x_D\right) = E\Psi^{(D)}\left(x_1, x_2, \ldots, x_D\right), \tag{6}$$

where $\Delta^{(D)}$ is the D-dimensional Laplacian and

$$r^2 = \sum_{i=1}^{D} x_i^2. \tag{7}$$

Transforming to the D-dimensional spherical coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{D-1})$, we separate variables using

$$\Psi^{(D)}\left(r, \theta_1, \theta_2, \ldots, \theta_{D-1}\right) = R_\ell^{(D)}\left(r\right)\ Y_\ell^{(D)}\left(\theta_1, \theta_2 \ldots, \theta_{D-1}\right), \tag{8}$$

where $Y_\ell^{(D)}\left(\theta_1, \theta_2 \ldots, \theta_{D-1}\right)$ is a normalized spherical harmonic with characteristic value $\ell\ (\ell\ +\ D\ -\ 2)$, $\ell\ =\ 0,\ 1,\ 2,\ \ldots$and $R_\ell^{(D)}\ (r)$ is a radial function that satisfies the equation:

$$\left\{-\frac{1}{2}\left[\frac{d^2}{dr^2} + \frac{D-1}{r}\frac{d}{dr} - \frac{\ell\left(\ell\ +\ D\ -\ 2\right)}{r^2}\right] + \frac{1}{2}r^2\right\} R_\ell^{(D)}(r) = ER_\ell^{(D)}\left(r\right) \tag{9}$$

Writing $R^{(D)}(r) = r^\ell e^{-r^2/2}F$, equation (9) gives

$$\frac{d^2F}{dr^2} + \left(\frac{D+2\ell-1}{r} - 2r\right)\frac{dF}{dr} + \left(2E - D - 2\ell\right)F = 0. \tag{10}$$

Changing the variable to $z = r^2$, we obtain

$$z\frac{d^2F}{dz^2}+\left(\ell+\frac{D}{2}-z\right)\frac{dF}{dz}-\frac{1}{2}\left(\ell+\frac{D}{2}-E\right)F=0. \tag{11}$$

Equation (11) is the well known Kummer's differential equation [46], whose regular solution at the origin is the confluent hypergeometric function

$$F={}_1F_1\left[\frac{1}{2}\left(\ell+\frac{D}{2}-E\right);\ell+\frac{D}{2};r^2\right]. \tag{12}$$

In order for the wave function to be square integrable, the hypergeometric series for the unconfined oscillator must terminate. This requirement is satisfied if there exists some non-negative integer n such that

$$E=2n+\ell+\frac{D}{2},\quad n=0,\,1,\,2,...\quad . \tag{13}$$

When the harmonic oscillator is enclosed in an impenetrable hypersphere of radius r_c, quantization results from the requirement that the radial wave function go to zero at r_c. The allowed energies are found when

$${}_1F_1\left[\frac{1}{2}\left(\ell+\frac{D}{2}-E\right);\ell+\frac{D}{2};r_c^2\right]=0, \tag{14}$$

where the successive roots are numbered $n=0,\,1,\,2,....$ As we mentioned previously, we found the allowed energies using Maple program. Our results for D=2, 3, 4 and 5 are reported in Tables II-V.

3 Power series method

The method that we will present has been used with much success in problems both with and without confinement. Some of the problems solved with this method are: the one-dimensional harmonic oscillator confined symmetrically and asymmetrically [39], the three-dimensional confined isotropic harmonic oscillator [42], the two-dimensional hydrogen atom confined in a circle with impenetrable walls [47] and the three-dimensional hydrogen atom confined in a hard sphere [10]. Other applications corresponding to free problems are: the hydrogen atom with a harmonic perturbation [48], the quartic harmonic oscillator and the double well potential for the inversion of NH_3 [49], in which the potential was represented by a polynomial of 20^{th} degree.

We will describe the method briefly. For further details see references [10], [39], [42], [47]. The Schrödinger equation (in natural units) for one degree of freedom for an arbitrary potential $V(x)$ can be written as

$$\psi''=2\left[V(x)-E\right]\quad for\ x<x_c \tag{15}$$

where x_c is the position of the impenetrable wall.

Now, we will suppose that the wavefunction is a function of the position x and of the energy E.

$$\psi=\psi(x,E). \tag{16}$$

Taking the partial derivative of equation (15) respect to the energy we obtain

$$\dot{\psi}''=2\left[V(x)-E\right]\dot{\psi}-2\psi \tag{17}$$

where $\dot{\psi}$ denotes partial differentiation with respect to the energy.

We need to obtain $\psi(x_c)$ and $\dot{\psi}(x_c)$. This is possible by making an initial guess E_j for the value of the energy and proceeding to integrate the equations (15) and (17). The corrected value of the energy is then obtained by means of the Newton-Raphson formula.

$$E_{j+1} = E_j - \frac{\psi(x_c, E_j)}{\dot{\psi}(x_c, E_j)} \quad . \tag{18}$$

With this new value E_{j+1}we calculate $\psi(x_c)$and $\dot{\psi}(x_c)$, and we use the formula (18) again to obtain a more precise value for the energy. We continue with this process until $|E_{n+1} - E_n| < \delta$, where δ is the desired accuracy for the calculation.

The integration of the equations (15) and (17) is achieved easily if we develop the wave function in a Taylor series around the origin, where we know the initial value of the wave function $\psi(0)$.

$$\psi(x) = \sum_p \frac{\psi^{(p)}(0)}{p!} x^p. \tag{19}$$

Defining

$$T_p = \frac{\psi^{(p)}(0)}{p!} x^p, \tag{20}$$

then

$$\psi(x) = \sum_p T_p. \tag{21}$$

We can also compute $\dot{\psi}(x)$ as follows

$$\dot{\psi}(x) = {}^{\partial\psi}/_{\partial E} = \sum_p {}^{\partial T_p}/_{\partial E} = \sum_p \dot{T}_p. \tag{22}$$

To make particular application of the method described above, we need to calculate the coefficients T_p and $\dot{T}_p$ for each dimension.

3.1 The one-dimensional confined harmonic oscillator

Substituting (21) in (1) the following recursion formula is obtained for the coefficients T_p

$$T_{p+2} = \frac{2(p + 1/2 - E)x^2}{(p+1)(p+2)} T_p \tag{23}$$

To obtain the recursion formula for the $\dot{T}_p$ coefficients, we take the partial derivative of equation (21) and use (23) to obtain

$$\dot{T}_{p+2} = \frac{2[(p + 1/2 - E)\,\dot{T}_p - T_p]x^2}{(p+1)(p+2)}. \tag{24}$$

For constructing the even states we used the initial conditions $\psi(0) = 1, \quad \psi'(0) = 0$, while for the odd states the initial conditions are $\psi(0) = 0, \quad \psi'(0) = 1$.

The other derivatives are obtained using the recurrence relationships for T_p and $\overset{\bullet}{T}_p$. The results obtained by this method are the same reported in Table I, those are improved results of a previous work [39]

3.2 The D-dimensional confined harmonic oscillator

Following similar steps as those described above, we found the recurrence relations for T_p and $\overset{\bullet}{T}_p$:

$$T_{p+2} = \frac{2\,(p+l+D/2-E)r^2}{(p+2)(p+2l+D)}T_p, \tag{25}$$

and

$$\overset{\bullet}{T}_{p+2} = \frac{2\,[(p+l+D/2-E)\,\overset{\bullet}{T}_p - T_p]r^2}{(p+2)(p+2l+D)}, \tag{26}$$

are obtained from Eq. (9).

The results obtained by this method are reported in Tables II–V. For $D = 3$ we show the improved results from reference [42].

4 Results and discussion

4.1 The one-dimensional confined harmonic oscillator

As we mentioned before, the first investigators who discussed the problem of the one-dimensional harmonic oscillator confined symmetrically in a box with impenetrable walls were Kothari and Auluck [17]–[18]. They found that the solutions of the Schrödinger equation could be written in terms of confluent hypergeometric functions. In order to obtain analytic approaches for the energies, they used expansions and approximations in the hypergeometric functions. In that way they obtained the correct qualitative behaviour of the energy levels and observed that the energy values increase quickly when diminishing the size of the box (for radius smaller than 4 au).

Baijal and Singh [25] followed a direct way and they obtained the energies numerically by finding the zeros of an equation equivalent to (5). Their numerical results were not accurate. The reasons of this failure are now clearly comprehensible; the absence of efficient algorithms and computation programs to evaluate the hypergeometric functions with high accuracy and the lack of computers that could execute these programs. Fortunately, these impediments have now been solved in a satisfactory way. In this work we used MAPLE, both for evaluation of the confluent hypergeometric functions and to find the roots of the set of equations (5), following the pioneer work of Baijal and Singh [25]. We used Maple's HYPERGEOM and FSOLVE functions and with a careful handling of the accuracy by means of the FULLDIGITS option, we were able to obtain the energy eigenvalues with an accuracy of 100 decimal digits.

On the other hand, by using the series method to this problem, and programming the respective equations in the UBASIC program by using real variables of 150 figures we calculate energy eigenvalues with 100 accuracy figures. Our results for the ground state are reported with 50 figures in Table I.

When comparing the energies obtained by both methods described above to 100 figures we found that the relative error between both calculations are smaller than 1×10^{-100}. This shows that the two quite different methods are both very stable and accurate. To our knowledge, these are the most precise calculations that have been reported until now.

4.2 The *2–D* and *3–D* confined harmonic oscillators

The exact formulation of the problem of the 2-D and 3-D confined harmonic oscillator was proposed by Aguilera-Navarro et al [40] in 1983. They obtained a transcendental equation in terms of confluent hypergeometric functions, whose roots are the energy eigenvalues, but they didn't solve it, noting that the numeric solution of that equation required hard computational effort. As in the one-dimensional problem, the researchers decided to use other approaches and methods such as the following: i) perturbation theory, ii) Padé approximants and iii) direct diagonalization of the Hamiltonan matrix in the basis set of the free particle, finding the matrix elements analytically. The results were obtained with 6 decimal places by diagonalizing matrixes up to 50x50. Those results were at that time considered to be the exact result.

Taseli and Zafer [41] used essentially the same method that Aguilera-Navarro et al, for studying the harmonic oscillator and polynomial potentials. Taseli and Zafer report their results with 30 decimal places. For comparison, their results are shown in Table II for the 2-D harmonic oscillator and in Table III for 3-D problem.

Our results obtained by solving equation (14) numerically using MAPLE and using the series method are reported with 50 decimal places for the ground state of the 2-D and 3-D confined harmonic oscillators in Table II and Table III respectively. The results obtained by these two methods coincided up to 100 significant figures.

4.3 The *4–D* and *5–D* confined harmonic oscillators.

Our results obtained by numerically solving the set of equations (14) by means of MAPLE program and by using the series method are reported with 50 significant figures, for the ground state of the 4-D and 5-D confined harmonic oscillators in Table IV and Table V respectively. The results obtained by these two methods coincide up to 100 decimal places. This is the first time that energy eigenvalues for the 4-D and 5-D confined harmonic oscillators have been reported.

5 Final remarks

Kotari and Auluck [17]–[18] formulated very early the problem in a correct way. Baijal and Singh [25] solved the problem numerically; however the accuracy in their results was limited to 1 part in 10,000. Due to the difficulties of obtaining the solution of this problem through the use of the hypergeometric functions, later researchers opted to use different methods and approaches. In this work we have shown that by using the hypergeometric functions of MAPLE computer algebra system, it is possible to obtain very accurate results. The accuracy of this method was confirmed by the series method described above. The coincidence of the results of both methods up to 100 decimal places shows that both approaches are very stable and accurate. The present results provide the benchmark to probe the accuracy of new methods.

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Table I. Ground state and first excited energies of the one-dimensional harmonic oscillator as a function of the box length $2x_c$. The energies are in units of $\hbar\omega$ and the distances in units of $\sqrt{\hbar/m\omega}$.

x_c	Ground state energy
0.5	4.9511293232 5413041195 1134080515 9857388997 9644551237
1.0	1.2984598320 3205669378 4230206450 2370269582 3439869300
1.5	0.6889317536 4684808297 9408577127 0594543020 2986219232
2.0	0.5374612092 8167516049 2498062617 3690881483 5739573494
2.5	0.5049541046 2368718050 9447174762 0176047282 7339258667
3.0	0.5003910829 2974859059 4328365218 4059730111 1218415624
3.5	0.5000180448 2030945381 6616486788 1801095234 3877637574
4.0	0.5000004908 5643052761 7123693880 1651645285 4015551420
4.5	0.5000000079 3815418303 1804233314 0697872385 1651045940
5.0	0.5000000000 7671713198 9118613783 6722860290 6861259474
6.0	0.5000000000 0000154791 6958282084 2671674954 5402080427
7.0	0.5000000000 0000000000 4098007117 5362147129 8519793926
8.0	0.5000000000 0000000000 0000001436 2707054755 7659037566
9.0	0.5000000000 0000000000 0000000000 0000670071 2965313260
10.0	0.5000000000 0000000000 0000000000 0000000000 0041764526
11.0	0.5000000000 0000000000 0000000000 0000000000 0000000000
x_c	First excited state energy
0.5	19.7745341792 0831989835 4604517172 0308265073 6578782343
1.0	5.0755820152 2678306601 7648991449 8809070971 4847782792
1.5	2.5049761785 3502402045 9213743876 6702242013 4355616840
2.0	1.7648164387 8063679020 2259586613 1246579254 5753601492
2.5	1.5514216545 5474477980 5797055037 4466896712 3648964272
3.0	1.5060815272 5279462165 2764212526 2481287731 9700177501
3.5	1.5003995211 9607101202 1916274934 7869410041 1346943413
4.0	1.5000146030 0712398734 9091703023 5066898625 6879252750
4.5	1.5000003041 6594363224 6948818889 9660548783 0443971120
5.0	1.5000000036 7158393112 6083763054 5056423792 2132790701
6.0	1.5000000000 0010821056 8920426294 9699221373 4008991988
7.0	1.5000000000 0000000039 3137796646 4980630573 4262919050
8.0	1.5000000000 0000000000 0000180898 7877745393 8765502196
9.0	1.5000000000 0000000000 0000000000 0107185446 9753799582
10.0	1.5000000000 0000000000 0000000000 0000000000 8268077088
11.0	1.5000000000 0000000000 0000000000 0000000000 0000000008

Table II. Energy eigenvalues for the two-dimensional isotropic confined harmonic oscillator for $n = 0, l = 0$ and $n = 0, l = 1$ as a function of the confinement radius r_c , and its comparison with Taseli's results[a] Ref.[41]. The energies are in units of $\hbar\omega$ and the distances in units of $\sqrt{\hbar/m\omega}$.

r_c	$n = 0, l = 0$
0.5	11.5936192506 8668479643 2170521897 6492016024 7818580170
1.0	3.0000000000 0000000000 0000000000 0000000000 0000000000
1.5	1.5235322602 5914873036 6271226432 8839654005 8580817118
2.0	1.1222085296 7891837492 4623606583 7385347311 4764534647
2.5	1.0199306851 0149764966 9029200736 9635571918 5241114989
3.0	1.0019367879 6432851707 9965303704 9698536351 6685928166
3.5	1.0001065838 7526319243 0778389006 5275059562 5162790712
4.0	1.0000033582 1521855746 5783438694 2835633641 8541234989
4.5	1.0000000616 1214749775 2604428805 1580090716 1945112286
5.0	1.0000000006 6534686266 8798296163 8488876290 9044905310
5.0[a]	1.0000000006 65385
6.0	1.0000000000 0001622254 6342645858 8254746724 7650899752
7.0	1.0000000000 0000000005 0308861850 7863587003 2026301379
7.0[a]	1.0000000000 0000000005 03
8.0	1.0000000000 0000000000 0000020202 7529579627 9788625777
9.0	1.0000000000 0000000000 0000000000 0010621779 8865980561
9.0[a]	1.0000000000 0000000000 0000000000 00
10.0	1.0000000000 0000000000 0000000000 0000000000 0736498307
11.0	1.0000000000 0000000000 0000000000 0000000000 0000000001
11.0[a]	1.0000000000 0000000000 0000000000 00
r_c	$n = 0, l = 1$
0.5	29.4056004466 9756111909 4989312607 2533606944 3885645082
1.0	7.5071721804 5194296125 5970569322 4683728514 7610899602
1.5	3.6322191884 0214310001 3125848888 3649489632 2914146137
2.0	2.4717752113 5017965087 7413015393 1937987026 1062386895
2.5	2.1057033473 6386137979 9168786765 6763723197 9306909584
3.0	2.0149671135 0308735580 2990773059 6826969574 0423889838
3.5	2.0011722745 5370550365 2986053259 5949083728 5949607461
4.0	2.0000497838 2870035813 8418130090 9885102064 2824637025
4.5	2.0000011782 3371178944 8218912905 1701308538 1618885601
5.0	2.0000000159 0394948216 9465718866 9200664238 5393456486
6.0	2.0000000000 0056676874 0232254818 0017888510 4212103477
7.0	2.0000000000 0000000241 2582416245 4898871780 6700297398
8.0	2.0000000000 0000000000 0001272098 8438202771 4243792590
9.0	2.0000000000 0000000000 0000000000 0849466212 9702927122
9.0[a]	2.0000000000 0000000000 000000000
10.0	2.0000000000 0000000000 0000000000 0000000007 2897978486
11.0	2.0000000000 0000000000 0000000000 0000000000 0000000081

Table III. Energy eigenvalues for the three-dimensional isotropic confined harmonic oscillator for $n = 0, l = 0$ and $n = 0, l = 1$ as a function of the confinement radius r_c, and its comparison with Taseli's results[b] Ref.[43]. The energies are in units of $\hbar\omega$ and the distances in units of $\sqrt{\hbar/m\omega}$.

r_c	$n = 0, l = 0$
0.5	19.7745341792 0831989835 4604517172 0308265073 6578782343
1.0	5.0755820152 2678306601 7648991449 8809070971 4847782792
1.5	2.5049761785 3502402045 9213743876 6702242013 4355616840
2.0	1.7648164387 8063679020 2259586613 1246579254 5753601492
2.5	1.5514216545 5474477980 5797055037 4466896712 3648964272
3.0	1.5060815272 5279462165 2764212526 2481287731 9700177501
3.5	1.5003995211 9607101202 1916274934 7869410041 1346943413
4.0	1.5000146030 0712398734 9091703023 5066898625 6879252750
4.5	1.5000003041 6594363224 6948818889 9660548783 0443971120
5.0	1.5000000036 7158393112 6083763054 5056423792 2132790701
5.0[b]	1.5000000036 715
6.0	1.5000000000 0010821056 8920426294 9699221373 4008991988
7.0	1.5000000000 0000000039 3137796646 4980630573 4262919050
7.0[b]	1.5000000000 0000000039 315
8.0	1.5000000000 0000000000 0000180898 7877745393 8765502196
9.0	1.5000000000 0000000000 0000000000 0107185446 9753799582
9.0[b]	1.5000000000 0000000000 000000000
10.0	1.5000000000 0000000000 0000000000 0000000000 8268077088
11.0	1.5000000000 0000000000 0000000000 0000000000 0000000008
11.0[b]	1.5000000000 0000000000 000000000
r_c	$n = 0, l = 1$
0.5	40.4282764968 8303569286 7657198337 4600503452 5299138678
1.0	10.2822569391 5401409565 3163129127 6677765073 6003398500
1.5	4.9035904194 0884107768 8890567799 6737146733 1869143993
2.0	3.2469470987 7100992231 2836277400 7282559791 2938650857
2.5	2.6881439638 9023726499 4772298426 8173993948 1291108167
3.0	2.5312924666 1555916863 7233915188 7007997989 8790276246
3.5	2.5029101642 9565984306 7531899942 2032477828 3713066610
4.0	2.5001437781 6983615678 2542969854 8358197775 8222780459
4.5	2.5000038701 0746391479 0391925903 5488977270 8091556844
5.0	2.5000000584 4093459377 8654916228 3700691294 3132687962
6.0	2.5000000000 0251914362 4008272991 8386092490 2462496073
7.0	2.5000000000 0000001256 5630089465 7702336239 3748249742
8.0	2.5000000000 0000000000 0007592671 4601199360 0262516226
9.0	2.5000000000 0000000000 0000000000 5714219098 1734768092
10.0	2.5000000000 0000000000 0000000000 0000000054 5548679893
11.0	2.5000000000 0000000000 0000000000 0000000000 0000000669

Table IV. Energy eigenvalues for the four-dimensional isotropic confined harmonic oscillator for $n = 0, l = 0$ and $n = 0, l = 1$ as a function of the confinement radius r_c . The energies are in units of $\hbar\omega$ and the distances in units of $\sqrt{\hbar/m\omega}$.

r_c	$n = 0, l = 0$
0.5	29.4056004466 9756111909 4989312607 2533606944 3885645082
1.0	7.5071721804 5194296125 5970569322 4683728514 7610899602
1.5	3.6322191884 0214310001 3125848888 3649489632 2914146137
2.0	2.4717752113 5017965087 7413015393 1937987026 1062386895
2.5	2.1057033473 6386137979 9168786765 6763723197 9306909584
3.0	2.0149671135 0308735580 2990773059 6826969574 0423889838
3.5	2.0011722745 5370550365 2986053259 5949083728 5949607462
4.0	2.0000497838 2870035813 8418130090 9885102064 2824637025
4.5	2.0000011782 3371178944 8218912905 1701308538 1618885601
5.0	2.0000000159 0394948216 9465718866 9200664238 5393456486
6.0	2.0000000000 0056676874 0232254818 0017888510 4212103477
7.0	2.0000000000 0000000241 2582416245 4898871780 6700297398
8.0	2.0000000000 0000000000 0001272098 8438202771 4243792590
9.0	2.0000000000 0000000000 0000000000 0849466212 9702927122
10.0	2.0000000000 0000000000 0000000000 0000000007 2897978486
11.0	2.0000000000 0000000000 0000000000 0000000000 0000000081
r_c	$n = 0, l = 1$
0.5	52.8003728316 3788346093 0950305785 0051346372 5631801073
1.0	13.3915380494 6480185532 0494009283 4067653436 0714152088
1.5	6.3173223844 6528516940 0939563655 0086913973 3614141779
2.0	4.0925993533 4910526414 2040215854 0583516233 3845977779
2.5	3.3030049737 7100747132 4786533205 4015439195 2504852928
3.0	3.0580504736 2975666805 2943138274 5161600133 0555173037
3.5	3.0063652100 0233518767 0490718817 2904327683 1566946907
4.0	3.0003661001 4285621959 4091284711 1602469471 7391623416
4.5	3.0000112193 8847681098 4494750875 7522162172 5417870407
5.0	3.0000001896 2579523225 9655764522 1335258633 9265148349
6.0	3.0000000000 0989077737 6549090900 3787454466 4191972235
7.0	3.0000000000 0000005781 9369976287 6503055609 2997874251
8.0	3.0000000000 0000000000 0040038694 5911963772 5227747959
9.0	3.0000000000 0000000000 0000000003 3961858787 5639028352
10.0	3.0000000000 0000000000 0000000000 0000000360 7297869310
11.0	3.0000000000 0000000000 0000000000 0000000000 0000004872

Table V. Energy eigenvalues for the five-dimensional isotropic confined harmonic oscillator for $n = 0, l = 0$ and $n = 0, l = 1$ as a function of the confinement radius r_c . The energies are in units of $\hbar\omega$ and the distances in units of $\sqrt{\hbar/m\omega}$.

r_c	$n = 0, l = 0$
0.5	40.4282764968 8303569286 7657198337 4600503452 5299138678
1.0	10.2822569391 5401409565 3163129127 6677765073 6003398500
1.5	4.9035904194 0884107768 8890567799 6737146733 1869143993
2.0	3.2469470987 7100992231 2836277400 7282559791 2938650857
2.5	2.6881439638 9023726499 4772298426 8173993948 1291108167
3.0	2.5312924666 1555916863 7233915188 7007997989 8790276246
3.5	2.5029101642 9565984306 7531899942 2032477828 3713066610
4.0	2.5001437781 6983615678 2542969854 8358197775 8222780459
4.5	2.5000038701 0746391479 0391925903 5488977270 8091556844
5.0	2.5000000584 4093459377 8654916228 3700691294 3132687963
6.0	2.5000000000 0251914362 4008272991 8386092490 2462496073
7.0	2.5000000000 0000001256 5630089465 7702336239 3748249742
8.0	2.5000000000 0000000000 0007592671 4601199360 0262516226
9.0	2.5000000000 0000000000 0000000000 5714219098 1734768092
10.0	2.5000000000 0000000000 0000000000 0000000054 5548679893
11.0	2.5000000000 0000000000 0000000000 0000000000 0000000669
r_c	$n = 0, l = 1$
0.5	66.4897565362 4517756305 3308770820 2330012102 2691586465
1.0	16.8277771096 2480257625 9617469161 0755246538 5604625452
1.5	7.8717304877 6674618813 0354341233 8621971714 2326103755
2.0	5.0100408656 3599642760 4228631968 4014661533 4471780425
2.5	3.9535289034 8568636015 5314458749 0430486562 2699519238
3.0	3.5982476989 7005161570 7481560771 7287260251 5760278006
3.5	3.5125803181 1996483612 8255886902 6750312869 7349417658
4.0	3.5008420738 2978240548 0678521299 6397537393 6306856269
4.5	3.5000294123 9950862348 5874703325 8737188786 0817955259
5.0	3.5000005567 1516742638 0149068727 2109923147 0235826666
6.0	3.5000000000 3515113957 7921310576 3564828467 8442048896
7.0	3.5000000000 0000024085 3639264861 3384517091 1914799496
8.0	3.5000000000 0000000000 0191153117 0090662417 7345620967
9.0	3.5000000000 0000000000 0000000018 2748879438 1689931231
10.0	3.5000000000 0000000000 0000000000 0000002159 5637174550
11.0	3.5000000000 0000000000 0000000000 0000000000 0000032115